Journal of Pure and Applied Algebra 214 (2010) 1193–1199

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Weighted limits in simplicial homotopy theory

Nicola Gambino *

Dipartimento di Matematica e Applicazioni, Università degli Studi di Palermo, via Archirafi 34, 90123 Palermo, Italy
School of Mathematics, The University of Manchester, Alan Turing Building, Oxford Road, Manchester M13 9PL, England, United Kingdom

ARTICLE INFO

Article history:
Received 12 March 2009
Received in revised form 26 July 2009
Available online 6 November 2009
Communicated by J. Adámek

MSC:
18G55
18D20
55U35
18G30

ABSTRACT

By combining ideas of homotopical algebra and of enriched category theory, we explain how two classical formulas for homotopy colimits, one arising from the work of Quillen and one arising from the work of Bousfield and Kan, are instances of general formulas for the derived functor of the weighted colimit functor.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

There are two classical formulas for the homotopy colimit of a diagram of simplicial sets $A : \mathbb{I} \to \mathbf{SSet}$. The first formula arises by considering the category **SSet** as equipped with the model structure, originally established by Quillen [19], for which the fibrant objects are exactly Kan complexes. The homotopy colimit of A is then expressed as the colimit

$$\lim_{\to} Q_{\mathrm{Proj}}(A), \tag{1}$$

where $Q_{\mathrm{Proj}}(A)$ denotes the cofibrant replacement of A with respect to the so-called projective model structure on the functor category $[\mathbb{I}, \mathbf{SSet}]$. This is the model structure for which the weak equivalences and the fibrations are defined as the natural transformations whose components are weak equivalences and fibrations in **SSet**, respectively. The second formula, which originates in [2], expresses the homotopy colimit of A as the coend

$$\int^{i\in\mathbb{I}} \mathrm{N}(i \downarrow \mathbb{I})^{\mathrm{op}} \times A_i, \tag{2}$$

where $\mathrm{N}(- \downarrow \mathbb{I})^{\mathrm{op}} : \mathbb{I} \to \mathbf{SSet}$ is the functor that maps $i \in \mathbb{I}$ into the nerve of the opposite of the coslice category $i \downarrow \mathbb{I}$. We refer to (1) as the *Quillen formula*, and to (2) as the *Bousfield–Kan formula* for homotopy colimits. The Quillen formula fits perfectly in the existing theory of Quillen adjunctions. Indeed, it can be seen as an instance of the general formula for the derived adjunction associated with a Quillen pair [14, Section 1.3.2]. This is because the projective model structure on $[\mathbb{I}, \mathbf{SSet}]$ is such that the functor sending a diagram to its colimit is a left Quillen functor. The Bousfield–Kan formula, however, does not seem to fit in the existing theory of Quillen adjunctions.

Our aim here is to fit both the Quillen formula and the Bousfield–Kan formula within the theory of Quillen adjunctions. To do so, we work with simplicial model categories, that is to say **SSet**-enriched categories equipped with a model structure that is suitably compatible with the model structure on **SSet**. Indeed, there are general forms of both the Quillen and the

* Corresponding address: Dipartimento di Matematica e Applicazioni, Università degli Studi di Palermo, via Archirafi 34, 90123 Palermo, Italy.
E-mail addresses: ngambino@math.unipa.it, nicola.gambino@gmail.com.

0022-4049/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.jpaa.2009.10.006

Bousfield–Kan formula for simplicial model categories [13, Chapter 18], which we want to include within our development. The key idea that allows us to achieve our goal is to consider not only the limit notions that are familiar from ordinary category theory, but also the more general limit notions known as weighted limits [16, Chapter 3]. We will establish that there are two ways of making the weighted colimit functor into a left Quillen functor in two variables. This result allows us to explain the presence of the two formulas discussed above. Indeed, the existence of two ways of regarding the weighted colimit functor as a left Quillen functor implies that there are two ways of computing its total left derived functor. One leads to the Quillen formula, and the other to the Bousfield–Kan formula.

To the best of the author's knowledge, this treatment of the Quillen formula and of the Bousfield–Kan formula for homotopy colimits does not appear in the existing literature. On the one hand, our approach differs from the one in [13, Chapter 18], where a general version of the Bousfield–Kan formula is assumed to be the homotopy colimit of a diagram by definition [13, Definition 18.1.2]. Here, instead, we derive a general version of the Quillen and Bousfield–Kan formula by combining our results with the general theory of derived adjunctions in the enriched setting [8,21]. Furthermore, while weighted limits are used only implicitly in [13, Chapter 18], they are exploited here as a fundamental concept. On the other hand, our approach differs also from the one taken in the literature on weighted limits in homotopy theory [1,10,11], which does not consider model structures. Here, as in [7], the combination of ideas of enriched category theory and of homotopical algebra plays instead an essential role. This is in a spirit similar to that of [22], which relates the formulas for homotopy limits involving the bar construction [18] with the abstract approach of homotopical categories [5].

Remark. The standard reference for enriched category theory is Kelly's book [16]. For the convenience of the reader, we will review the notion of a weighted limit in the special case of simplicial categories. For the theory of model categories, we refer the reader to Hovey's book [14]. For further information concerning homotopy limits, the reader is invited to refer also to [2,6,9,12,23]. General approaches to homotopy limits are developed in [3,4,20].

2. Simplicial model categories

We write **SSet** for the category of simplicial sets. The category **SSet** will always be considered here as equipped with Quillen's model structure [19], which can be established not only using the geometric realization functor [14, Chapter 3], but also in a purely combinatorial way [15]. Finite products determine a monoidal structure on **SSet** that satisfies the axioms for a monoidal model category [14, Proposition 4.28]. The internal function space makes **SSet** into a monoidal closed category. For $X, Y \in \mathbf{SSet}$, we write $\mathbf{SSet}(X, Y)$ for their internal function space.

By a *simplicial category* we mean a category enriched in **SSet**. If A and B are objects of a simplicial category $\mathbb{C}$, we write $\mathbb{C}(A, B)$ for the simplicial set of maps from A to B. As a special case of the general concepts of enriched category theory [16, Section 1.2], we obtain the notions of a simplicial functor and of a simplicial natural transformation. These notions give rise to the 2-category **SCat** of simplicial categories, simplicial functors, and simplicial natural transformations. As a special case of the construction described in [16, Section 1.3], each simplicial category $\mathbb{C}$ has an associated *underlying category*, with the same objects as $\mathbb{C}$ and with maps $f : A \to B$ given by the 0-simplices of $\mathbb{C}(A, B)$. The function assigning to a simplicial category its underlying category extends to a 2-functor $\mathbf{SCat} \to \mathbf{Cat}$, where **Cat** is the 2-category of locally small categories, functors, and natural transformations. The category **SSet** can be regarded as a simplicial category, with enrichment given by its internal function space.

We recall in Definition 2.1 the notion of a *simplicial model category*. For this, we need to introduce some notation. For a simplicial category $\mathbb{C}$, a pair of maps $f : A \to B$ and $g : C \to D$ in $\mathbb{C}$ determines the following commutative diagram in **SSet**

$$\begin{tikzcd}
\mathbb{C}(B, C) \arrow[r, "\mathbb{C}(B,g)"] \arrow[d, "\mathbb{C}(f,C)"'] & \mathbb{C}(B, D) \arrow[d, "\mathbb{C}(f,D)"] \\
\mathbb{C}(A, C) \arrow[r, "\mathbb{C}(A,g)"'] & \mathbb{C}(A, D) .
\end{tikzcd}$$

We write $\mathbb{C}(f, g) : \mathbb{C}(B, C) \to \mathbb{C}(A, D)$ for the common value of the composites of the diagram above. Since **SSet** has pullbacks, we obtain a canonical map

$$[f, g] : \mathbb{C}(B, C) \to \mathbb{C}(A, C) \times_{\mathbb{C}(A,D)} \mathbb{C}(B, D).$$

This map is used in the next definition, which is essentially due to Quillen [19, Chapter II] and is a special case of the general notion of an enriched model category [8, Definition 3.3]. See also [14, Definition 4.2.18].

Definition 2.1. A *simplicial model structure* on a simplicial category $\mathbb{C}$ is a model structure on the underlying category of $\mathbb{C}$ such that condition $(*)$ holds.

$(*)$ If $f : A \to B$ is a cofibration and $g : C \to D$ is a fibration in $\mathbb{C}$, then the map $[f, g]$ is a fibration in **SSet** which is also a weak equivalence whenever either f or g is so.

A *simplicial model category* is a simplicial category that is equipped with a simplicial model structure.

Definition 2.2 exploits the fact that a simplicial adjunction between simplicial categories, that is to say an adjunction in **SCat**, is mapped by the 2-functor **SCat** $\to$ **Cat** into an adjunction of ordinary categories.

Definition 2.2. A *simplicial Quillen adjunction* between simplicial model categories is a simplicial adjunction whose underlying adjunction is a Quillen adjunction.

We will need also a counterpart of the notion of Quillen adjunction in two variables [14, Definition 4.2.1] in the simplicially enriched setting. For this, recall from [16, Section 1.4] that the 2-category **SCat** inherits a cartesian structure from the category **SSet**. A simplicial functor of the form $\Phi : \mathbb{C} \times \mathbb{D} \to \mathbb{E}$ will be referred to as a *simplicial functor in two variables*. Given a simplicial functor $\Phi : \mathbb{C} \times \mathbb{D} \to \mathbb{E}$, for $f : A \to B$ in $\mathbb{C}$ and $g : C \to D$ in $\mathbb{D}$ we write $\Phi(f, g) : \Phi(A, C) \to \Phi(B, D)$ for the common value of the composites in the commutative diagram

$$\begin{tikzcd} \Phi(A, C) \arrow[r, "\Phi(f,C)"] \arrow[d, "\Phi(A,g)"'] & \Phi(B, C) \arrow[d, "\Phi(B,g)"] \\ \Phi(A, D) \arrow[r, "\Phi(f,D)"'] & \Phi(B, D) . \end{tikzcd}$$

When $\mathbb{E}$ has pushouts, the commutativity of the diagram determines a canonical map

$$\langle f, g\rangle : \Phi(A, D) \sqcup_{\Phi(A,C)} \Phi(B, C) \to \Phi(B, D).$$

We use this map in the next definition.

Definition 2.3. Let $\mathbb{C}, \mathbb{D}, \mathbb{E}$ be simplicial categories whose underlying categories are equipped with model structures. A simplicial functor in two variables $\Phi : \mathbb{C} \times \mathbb{D} \to \mathbb{E}$ is said to be a *left Quillen functor in two variables* if the following conditions hold.

(i) Φ is cocontinuous in each variable.
(ii) If $f : A \to B$ is a cofibration in $\mathbb{C}$ and $g : C \to D$ is a cofibration in $\mathbb{D}$, then $\langle f, g\rangle$ is a cofibration in $\mathbb{E}$, which is also a weak equivalence whenever either f or g is so.

We say that a simplicial functor $\Phi : \mathbb{C} \times \mathbb{D} \to \mathbb{E}$ is a *right Quillen functor in two variables* if its dual $\Phi^{\mathrm{op}} : \mathbb{C}^{\mathrm{op}} \times \mathbb{D}^{\mathrm{op}} \to \mathbb{E}^{\mathrm{op}}$ is a left Quillen functor in two variables.

Our study of homotopy limits in simplicial model categories involves examples of the general situation isolated in Definition 2.4. Recall that a simplicial adjunction in two variables consists of simplicial functors

$$\Phi : \mathbb{C} \times \mathbb{D} \to \mathbb{E}, \qquad \Theta : \mathbb{D}^{\mathrm{op}} \times \mathbb{E} \to \mathbb{C}, \qquad \Psi : \mathbb{C}^{\mathrm{op}} \times \mathbb{E} \to \mathbb{D},$$

and simplicial natural isomorphisms, for $C \in \mathbb{C}$, $D \in \mathbb{D}$, and $E \in \mathbb{E}$,

$$\mathbb{C}\big(C, \Theta(D, E)\big) \cong \mathbb{E}\big(\Phi(C, D), E\big) \cong \mathbb{D}\big(D, \Psi(C, E)\big) .$$

In these circumstances, Φ is a left adjoint in two variables, Ψ and Θ are right adjoints in two variables. Enriched adjunctions in two variables have been studied in connection with homotopy limits in [10]. The following definition is the simplicially enriched counterpart of the notion of a Quillen adjunction in two variables [14, Definition 4.2.1].

Definition 2.4. A *simplicial Quillen adjunction in two variables* is a simplicial adjunction in two variables (Φ, Θ, Ψ) such that the following equivalent conditions hold:

(i) $\Phi : \mathbb{C} \times \mathbb{D} \to \mathbb{E}$ is a left Quillen functor in two variables,
(ii) $\Theta : \mathbb{D}^{\mathrm{op}} \times \mathbb{E} \to \mathbb{C}$ is a right Quillen functor in two variables,
(iii) $\Psi : \mathbb{C}^{\mathrm{op}} \times \mathbb{E} \to \mathbb{C}$ is a right Quillen functor in two variables.

We conclude this section by providing an example of Quillen adjunction in two variables which is going to be useful in Section 3. The example involves the notions of *tensor* and *cotensor*, which we recall from [16, Section 3.7]. Let $\mathbb{C}$ be a simplicial category. The existence of tensors in $\mathbb{C}$ can be expressed as the existence, for every $A \in \mathbb{C}$, of a simplicial adjunction of the form

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift right, "\mathbb{C}(A,-)"'] \arrow[r, phantom, "\perp"] & \mathbf{SSet} . \arrow[l, shift right, "(-)\otimes A"'] \end{tikzcd}$$

Here, the left adjoint maps $X \in \mathbf{SSet}$ into $X \otimes A$, the *X-tensor of A*, which is characterized by the existence of a simplicial natural isomorphism with components

$$\mathbb{C}\big(X \otimes A, B\big) \cong \mathbf{SSet}\big(X, \mathbb{C}(A, B)\big). \tag{3}$$

Cotensors are defined dually: to say that $\mathbb{C}$ has *cotensors* is to say that for every $B \in \mathbb{C}$ there exists a simplicial adjunction of the form

$$\begin{tikzcd} \mathbb{C}^{\mathrm{op}} \arrow[r, shift right, "\mathbb{C}(-,B)"'] \arrow[r, phantom, "\perp"] & \mathbf{SSet}. \arrow[l, shift right, "{[-,B]}"'] \end{tikzcd}$$

The left adjoint maps $X \in \mathbf{SSet}$ into $[X, B] \in \mathbb{C}$, the *X-cotensor of B*, which is characterized by the existence of a simplicial natural isomorphism with components

$$\mathbb{C}\big(A, [X, B]\big) \cong \mathbf{SSet}\big(X, \mathbb{C}(A, B)\big). \tag{4}$$

When $\mathbb{C}$ has both tensors and cotensors, we have a simplicial adjunction in two variables involving the functors

$$\begin{aligned} &\Phi : \mathbf{SSet} \times \mathbb{C} \to \mathbb{C}, && \Phi(X, A) =_{\mathrm{def}} X \otimes A, \\ &\Theta : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{SSet}, && \Theta(A, B) =_{\mathrm{def}} \mathbb{C}(A, B), \\ &\Psi : \mathbf{SSet}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}, && \Psi(X, B) =_{\mathrm{def}} [X, B]. \end{aligned}$$

The following lemma is exploited repeatedly in Section 3.

Lemma 2.5. *Let $\mathbb{C}$ be a tensored and cotensored simplicial category, and assume that its underlying category is equipped with a model structure. The following conditions are equivalent:*

(i) *the functor $(-) \otimes (-) : \mathbf{SSet} \times \mathbb{C} \to \mathbb{C}$ is a left Quillen functor in two variables,*
(ii) *the functor $\mathbb{C}(-, -) : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{SSet}$ is a right Quillen functor in two variables,*
(iii) *the functor $[-, -] : \mathbf{SSet}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ is a right Quillen functor in two variables.*

These conditions hold if and only if $\mathbb{C}$ is a simplicial model category.

Proof. The equivalence of the conditions is an instance of the equivalence between the conditions in Definition 2.4. The very definition of a simplicial model category, given in Definition 2.1, simply restates condition (ii). □

When regarded as a simplicial category, **SSet** admits both tensors and cotensors, which are given by the cartesian product and the internal function spaces, respectively. Again, this is a special case of a general fact in enriched category theory [16, Section 3.7].

3. Homotopy limits

Let $\mathbb{C}$ be a simplicial category. For a small simplicial category $\mathbb{I}$, we write $[\mathbb{I}, \mathbb{C}]$ for the simplicial category whose underlying category has simplicial functors from $\mathbb{I}$ to $\mathbb{C}$ as objects and simplicial natural transformations as maps. We often refer to functors $A : \mathbb{I} \to \mathbb{C}$ as *diagrams*. If $\mathbb{C}$ is equipped with a simplicial model structure, there are at least two possible simplicial model structures on $[\mathbb{I}, \mathbb{C}]$, which are generally referred to as the *projective* and *injective* model structures. To define them, we need to introduce some terminology. A simplicial natural transformation $f : A \to B$ is said to be a *pointwise weak equivalence* if each of its components $f_i : A_i \to B_i$, for $i \in \mathbb{I}$, is a weak equivalence. The notions of a *pointwise fibration* and of a *pointwise cofibration* are defined analogously. The lifting properties in Definition 3.1 below always refer to commutative diagrams and fillers in the underlying category of $[\mathbb{I}, \mathbb{C}]$.

Definition 3.1. Let $f : A \to B$ be a simplicial natural transformation.

(i) We say that f is a *projective cofibration* if it has the left lifting property with respect to the simplicial natural transformations which are pointwise acyclic fibrations.
(ii) We say that f is a *injective fibration* if it has the right lifting property with respect to the simplicial natural transformations which are pointwise acyclic cofibrations.

The projective model structure is defined as follows:

$$[\mathbb{I}, \mathbb{C}]_{\mathrm{Proj}} \begin{cases} \text{weak equivalences} = \text{pointwise weak equivalences}, \\ \text{fibrations} = \text{pointwise fibrations}, \\ \text{cofibrations} = \text{projective cofibrations}. \end{cases}$$

The cofibrant objects of the projective model structure will be referred to as the *projectively cofibrant* diagrams. We do not need to introduce special terminology for the fibrant objects, since a diagram is fibrant in the projective model structure if and only if it is pointwise fibrant. The fibrant and cofibrant replacement of a diagram A with respect to the projective model structure will be denoted as $R_{\mathrm{Proj}}(A)$ and $Q_{\mathrm{Proj}}(A)$, respectively. Note that $R_{\mathrm{Proj}}(A)$ can be defined with the fibrant replacement of $\mathbb{C}$, provided that this is a simplicial functor. The injective model structure is defined dually, as follows:

$$[\mathbb{I}, \mathbb{C}]_{\mathrm{Inj}} \begin{cases} \text{weak equivalences} = \text{pointwise weak equivalences}, \\ \text{fibrations} = \text{injective fibrations}, \\ \text{cofibrations} = \text{pointwise cofibrations}. \end{cases}$$

There is an evident notion of *injectively fibrant* diagram. The cofibrant objects in the injective model structure are instead the pointwise cofibrant diagrams. The fibrant and cofibrant replacements of a diagram A with respect to the injective model structure are denoted as $R_{\mathrm{Inj}}(A)$ and $Q_{\mathrm{Inj}}(A)$, respectively.

For if $\mathbb{C}$ is **SSet**, the projective model structure was established by Quillen [19] and the injective model structure by Heller [12]. A general result of Lurie [17, Proposition A.3.3.2] isolates conditions that guarantee the existence of projective and injective model structures on simplicial categories. When they exist, the projective and the injective model category are Quillen equivalent [17, Proposition A.3.3.8] and satisfy the axioms for a simplicial model category, as a simple calculation shows. From now on, when we refer to these model structures, we implicitly assume their existence. Indeed, our focus is not on the conditions that ensure the existence of these model structures, but rather on how their existence allows us to study the homotopical behaviour of limit functors. Similarly, when we refer to limits and colimits, we tacitly assume their existence.

Since simplicial categories are enriched categories, they admit not only standard limit notions, but also notions of weighted limit, which we recall briefly from [16, Chapter 3]. Since limits and colimits are dual notions, it suffices to study one of them. We study colimits. When treating weighted colimits, a *weight* is a functor $X : \mathbb{I}^{\mathrm{op}} \to \mathbf{SSet}$. The existence of weighted colimits in a simplicial category $\mathbb{C}$ can be expressed as the existence, for every diagram A, of a simplicial adjunction of the form

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift right=1.5ex, "\mathbb{C}(A_{(-)},-)"'] \arrow[r, phantom, "\perp"] & {[\mathbb{I}^{\mathrm{op}}, \mathbf{SSet}]} \arrow[l, shift right=1.5ex, "(-)\otimes_{\mathbb{I}} A"'] \end{tikzcd} .$$

The left adjoint sends a weight X to $X \otimes_{\mathbb{I}} A$, the *X-weighted colimit of A*, which is characterized by the existence of a simplicial natural isomorphism with components

$$\mathbb{C}(X \otimes_{\mathbb{I}} A, B) \cong [\mathbb{I}^{\mathrm{op}}, \mathbf{SSet}]\big(X_{(-)}, \mathbb{C}(A_{(-)}, B)\big). \tag{5}$$

We think of $X \otimes_{\mathbb{I}} A$ as an $\mathbb{I}$-indexed tensor, with the isomorphism in (3) being analogous to that in (5). Indeed, when $\mathbb{I}$ is the terminal simplicial category $\mathbf{1}$, weighted colimits reduce to tensors. This point of view is supported by the following formula [16, Section 3.10], which expresses weighted colimits in terms of tensors and coends:

$$X \otimes_{\mathbb{I}} A \cong \int^{i \in \mathbb{I}} X_i \otimes A_i. \tag{6}$$

When $\mathbb{C}$ is cotensored, the existence of X-weighed colimits is equivalent to the existence of a simplicial adjunction of the form

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift right=1.5ex, "{[X_{(-)}, \Delta(-)]}"'] \arrow[r, phantom, "\perp"] & {[\mathbb{I}, \mathbb{C}]} \arrow[l, shift right=1.5ex, "X\otimes_{\mathbb{I}}(-)"'] \end{tikzcd} .$$

Here, the right adjoint, which maps $A \in \mathbb{C}$ into the constant diagram sending an object $i \in \mathbb{I}$ into the cotensor $[X_i, A] \in \mathbb{C}$, should be understood as a weighted analogue of the diagonal functor that participates in the adjunction expressing the existence of colimits in an ordinary category. As shown in [16, Section 3.9], the colimit of a simplicial functor $A : \mathbb{I} \to \mathbf{SSet}$ can be expressed as a weighted colimit via the isomorphism

$$\varinjlim A \cong 1 \otimes_{\mathbb{I}} A, \tag{7}$$

where $1 : \mathbb{I}^{\mathrm{op}} \to \mathbf{SSet}$ denotes the weight with constant value the terminal object of **SSet**.

If $\mathbb{C}$ admits cotensors and weighted colimits, the weighted colimit functor

$$(-) \otimes_{\mathbb{I}} (-) : [\mathbb{I}^{\mathrm{op}}, \mathbf{SSet}] \times [\mathbb{I}, \mathbb{C}] \to \mathbb{C}$$

is part of the simplicial adjunction in two variables which involves the following simplicial functors:

$$\Phi : [\mathbb{I}^{\mathrm{op}}, \mathbf{SSet}] \times [\mathbb{I}, \mathbb{C}] \to \mathbb{C}, \qquad \Phi(X, A) =_{\mathrm{def}} X \otimes_{\mathbb{I}} A, \tag{8}$$

$$\Theta : [\mathbb{I}, \mathbb{C}]^{\mathrm{op}} \times \mathbb{C} \to [\mathbb{I}^{\mathrm{op}}, \mathbf{SSet}], \qquad \Theta(A, B) =_{\mathrm{def}} \mathbb{C}(A_{(-)}, B), \tag{9}$$

$$\Psi : [\mathbb{I}^{\mathrm{op}}, \mathbf{SSet}]^{\mathrm{op}} \times \mathbb{C} \to [\mathbb{I}, \mathbb{C}], \qquad \Psi(X, B) =_{\mathrm{def}} [X_{(-)}, B]. \tag{10}$$

Theorems 3.2 and 3.3 show that there are two choices of model structures that allow us to regard this simplicial adjunction in two variables as a Quillen adjunction. In particular, there will be two ways of regarding the weighted colimit functor as a left Quillen functor in two variables. The proofs of Theorems 3.2 and 3.3 refer to the functors Φ, Θ, Ψ defined in (8)–(10), respectively.

Theorem 3.2. *Let $\mathbb{C}$ be a simplicial model category. If we consider the category of weights $[\mathbb{I}^{\mathrm{op}}, \mathbf{SSet}]$ as equipped with the injective model structure and the category of diagrams $[\mathbb{I}, \mathbb{C}]$ as equipped with the projective model structure, then the weighted colimit functor is a left Quillen functor in two variables.*

Proof. We need to show that

$$\Phi : [\mathbb{I}^{op}, \mathbf{SSet}]_{Inj} \times [\mathbb{I}, \mathbb{C}]_{Proj} \to \mathbb{C}$$

is a left Quillen functor in two variables. The following conditions are equivalent:

(i) $\Phi : [\mathbb{I}^{op}, \mathbf{SSet}]_{Inj} \times [\mathbb{I}, \mathbb{C}]_{Proj} \to \mathbb{C}$ is a left Quillen functor,
(ii) $\Theta : [\mathbb{I}, \mathbb{C}]^{op}_{Proj} \times \mathbb{C} \to [\mathbb{I}^{op}, \mathbf{SSet}]_{Inj}$ is a right Quillen functor,
(iii) $\Psi : [\mathbb{I}^{op}, \mathbf{SSet}]^{op}_{Inj} \times \mathbb{C} \to [\mathbb{I}, \mathbb{C}]_{Proj}$ is a right Quillen functor.

The assumption that $\mathbb{C}$ is a simplicial model category and Lemma 2.5 imply that the cotensor functor $[-, -] : \mathbf{SSet}^{op} \times \mathbb{C} \to \mathbb{C}$ is a right Quillen functor in two variables. This implies that condition (iii) holds. □

There is a second choice of Quillen model structures that allows us to make the weighted colimit functor into a left Quillen functor in two variables.

Theorem 3.3. *Let $\mathbb{C}$ be a simplicial model category. If we consider the category of weights $[\mathbb{I}^{op}, \mathbf{SSet}]$ as equipped with the projective model structure and the category of diagrams $[\mathbb{I}, \mathbb{C}]$ as equipped with the injective model structure, then the weighted colimit functor is a left Quillen functor in two variables.*

Proof. We need to show that

$$\Phi : [\mathbb{I}^{op}, \mathbf{SSet}]_{Proj} \times [\mathbb{I}, \mathbb{C}]_{Inj} \to \mathbb{C}$$

is a left Quillen functor in two variables. The following conditions are equivalent:

(i) $\Phi : [\mathbb{I}^{op}, \mathbf{SSet}]_{Proj} \times [\mathbb{I}, \mathbb{C}]_{Inj} \to \mathbb{C}$ is a left Quillen functor,
(ii) $\Theta : [\mathbb{I}, \mathbb{C}]^{op}_{Inj} \times \mathbb{C} \to [\mathbb{I}^{op}, \mathbf{SSet}]_{Proj}$ is a right Quillen functor,
(iii) $\Psi : [\mathbb{I}^{op}, \mathbf{SSet}]^{op}_{Proj} \times \mathbb{C} \to [\mathbb{I}, \mathbb{C}]_{Inj}$ is a right Quillen functor.

It is possible to show that condition (ii) holds using the fact that $\mathbb{C}(-, -) : \mathbb{C}^{op} \times \mathbb{C} \to \mathbf{SSet}$ is a right Quillen functor in two variables, analogously to the way Theorem 3.2 was proved. □

There are analogous results for weighted limits. These assert that there are two ways of making the weighted limit functor into a right Quillen functor in two variables. A first possibility is to consider both the category of diagrams and the category of weights as equipped with the injective model structure; a second possibility is to consider both the category of diagrams and the category of weights as equipped with the projective model structure.

4. Derived functors

Theorems 3.2 and 3.3 allow us to apply the theory of derived functors in the enriched setting, as developed in [8], and deduce the existence of the total derived functor of the weighted colimit functor, and to provide explicit expressions for it. Of course, there is also an analogous development for homotopy limits, which we do not spell out for brevity.

We write $\mathrm{Ho}(\mathbf{SSet})$ for the homotopy category of $\mathbf{SSet}$, and $\mathrm{Ho}(\mathbb{C})$ for the homotopy category of a simplicial model category $\mathbb{C}$, which is a $\mathrm{Ho}(\mathbf{SSet})$-enriched category by the results in [8,14,21]. Let us consider the total left derived functor of the weighted limit functor

$$(-) \otimes^{L}_{\mathbb{I}} (-) : \mathrm{Ho}[\mathbb{I}^{op}, \mathbf{SSet}] \times \mathrm{Ho}[\mathbb{I}, \mathbb{C}] \to \mathrm{Ho}(\mathbb{C}).$$

The existence of two ways of making the weighted colimit functor into a left Quillen functor in two variables means that there are two different, but equivalent, formulas for computing its total left derived functor. The first formula arises from considering the choice of model structures in Theorem 3.2. This gives the following expression for the left derived functor:

$$X \otimes^{L}_{\mathbb{I}} A = X \otimes_{\mathbb{I}} Q_{Proj}(A). \tag{11}$$

This formula is the result of a simplification from $Q_{Inj}(X) \otimes_{\mathbb{I}} Q_{Proj}(A)$, which would be the general formula for the derived functor. This simplification is possible because being cofibrant in the injective model structure on $[\mathbb{I}^{op}, \mathbf{SSet}]$ means being pointwise cofibrant, which is satisfied by any weight since every object is cofibrant in $\mathbf{SSet}$ [14, Proposition 3.2.2]. By the formula in (7), the homotopy colimit functor

$$\underrightarrow{\mathrm{holim}} : \mathrm{Ho}[\mathbb{I}, \mathbb{C}] \to \mathrm{Ho}(\mathbb{C})$$

can be defined as mapping a diagram A into $1 \otimes^{L}_{\mathbb{I}} A$, where $1 : \mathbb{I}^{op} \to \mathbf{SSet}$ denotes the weight with constant value the terminal object of $\mathbf{SSet}$. By (7) and (11), we obtain the following formula for homotopy colimits:

$$\underrightarrow{\mathrm{holim}}(A) \cong \lim_{\rightarrow} Q_{Proj}(A).$$

This is a generalized version of the Quillen formula in (1). Indeed, it arises also by considering the projective model structure on the category $[\mathbb{I}, \mathbb{C}]$, so that the colimit functor becomes a left Quillen functor in the familiar adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift right, "\Delta"'] \arrow[r, phantom, "\perp"] & {[\mathbb{I}, \mathbb{C}]}. \arrow[l, shift right, "\varinjlim"'] \end{tikzcd}$$

The second formula arises by considering the choice of model structures given in Theorem 3.3. This gives the following expression for homotopy colimits:

$$X \otimes^{\mathrm{L}}_{\mathbb{I}} A = Q_{\mathrm{Proj}}(X) \otimes_{\mathbb{I}} Q_{\mathrm{Inj}}(A). \tag{12}$$

If X is the constant weight $1 : \mathbb{I}^{\mathrm{op}} \to \mathbf{SSet}$, we have

$$\underrightarrow{\mathrm{holim}}(A) = Q_{\mathrm{Proj}}(1) \otimes_{\mathbb{I}} Q_{\mathrm{Inj}}(A). \tag{13}$$

This is a generalized version of the Bousfield–Kan formula given in (2). We expand the formula (13) in two steps. First, we express the weighted colimit in (13) as a coend using the formula in (6). Secondly, we exploit [13, Proposition 14.8.8] and consider the functor $\mathrm{N}(- \downarrow \mathbb{I})^{\mathrm{op}} : \mathbb{I}^{\mathrm{op}} \to \mathbf{SSet}$, defined in Section 1, as the projective cofibrant replacement for the constant weight $1 : \mathbb{I}^{\mathrm{op}} \to \mathbf{SSet}$. We obtain

$$\underrightarrow{\mathrm{holim}}(A) \cong \int^{i \in \mathbb{I}} \mathrm{N}(i \downarrow \mathbb{I})^{\mathrm{op}} \otimes \big(Q_{\mathrm{Inj}}(A)\big)_i. \tag{14}$$

When A is pointwise cofibrant, it is cofibrant in the injective model structure, and therefore we have

$$\underrightarrow{\mathrm{holim}}(A) \cong \int^{i \in \mathbb{I}} \mathrm{N}(i \downarrow \mathbb{I})^{\mathrm{op}} \otimes A_i\,. \tag{15}$$

To obtain the formula in (2), it suffices to consider (14) in the special case of $\mathbb{C} = \mathbf{SSet}$. In this situation, the requirement that A is pointwise cofibrant is always satisfied [14, Proposition 3.2.2], so we reduce to (15). Finally, since $\mathbb{C} = \mathbf{SSet}$, the tensor in (15) becomes the cartesian product, so (2) is indeed a special case of (13).

Remark. The formula in (15) coincides with the definition of the homotopy colimit given in [13, Definition 18.1.2] and [13, Example 18.3.6]. In [13, Definition 18.1.2] the formula in (15) is assumed as the definition of the homotopy colimit also when the diagram A is not pointwise cofibrant. However, properties of homotopy invariance are proved under the assumption that A satisfies the hypothesis of being pointwise cofibrant [13, Theorem 18.5.2 and Theorem 18.5.3].

Acknowledgements

I am grateful to André Joyal, Michael Mandell, Vincent Schmitt, and Myles Tierney for helpful discussions. The support of the Laboratoire de Combinatoire et d'Informatique Mathématique of the Université du Québec à Montréal and of the Centre de Recerca Matemàtica is gratefully acknowledged.

References

[1] D. Bourn, J.-L. Cordier, A general formulation of homotopy limits, Journal of Pure and Applied Algebra 29 (2) (1989) 129–141.
[2] A.K. Bousfield, D.M. Kan, Homotopy Limits, Completions, and Localizations, in: Lecture Notes in Mathematics, vol. 304, Springer, 1972.
[3] W. Chachólski, J. Scherer, Homotopy theory of diagrams, Memoirs of the American Mathematical Society 155 (736) (2002).
[4] D.-C. Cisinski, Catégories dérivables. Available from the author's web page, 2003.
[5] W.G. Dwyer, P.S. Hirschhorn, D.M. Kan, J.H. Smith, Homotopy Limit Functors on Model Categories and Homotopical Categories, in: Mathematical Surveys and Monographs, vol. 113, American Mathematical Society, 2004.
[6] W.G. Dwyer, J. Spalinski, Homotopy theories and model categories, in: I.M. James (Ed.), Handbook of algebraic topology, Elsevier, 1995, pp. 73–126.
[7] N. Gambino, Homotopy limits for 2-categories, Mathematical Proceedings of the Cambridge Philosophical Society 145 (1) (2008) 43–63.
[8] L. Gaunce Lewis Jr., M.A. Mandell, Modules in monoidal model categories, Journal of Pure and Applied Algebra 210 (2007) 395–421.
[9] P.G. Goerss, J.F. Jardine, Simplicial Homotopy Theory, in: Progress in Mathematics, vol. 174, Birkäuser, 1999.
[10] J.W. Gray, Closed categories, lax limits, and homotopy limits, Journal of Pure and Applied Algebra 19 (1980) 127–158.
[11] J.W. Gray, The representation of limits, lax limits and homotopy limits as sections, in: Mathematical Applications of Category Theory (Denver, Col., 1983), in: Contemporary Mathematics, vol. 30, American Mathematical Society, 1984, pp. 63–83.
[12] A. Heller, Homotopy theories, Memoirs of the American Mathematical Society 71 (383) (1988).
[13] P.S. Hirschhorn, Model Categories and their Localizations, in: Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, 2002.
[14] M. Hovey, Model Categories, in: Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, 1998.
[15] A. Joyal, M. Tierney, An introduction to simplicial homotopy theory, Hopf Archive (1999).
[16] G.M. Kelly, Basic Concepts of Enriched Category Theory, in: London Mathematical Society Lecture Note, vol. 64, Cambridge University Press, 1982, Available online in the series Reprints in Theory and Applications of Categories.
[17] J. Lurie, Higher topos theory. ArXiv: math/060804v4. 2008.
[18] J.P. May, The Geometry of Iterated Loop Spaces, in: Lecture Notes in Mathematics, vol. 272, Springer, 1972.
[19] D.G. Quillen, Homotopical Algebra, in: Lecture Notes in Mathematics, vol. 43, Springer, 1967.
[20] A. Radulescu-Banu, Cofibrations in homotopy theory, Hopf Archive, (2006).
[21] S. Schwede, B. Schipley, Algebras and modules in monoidal model categories, Proceedings of the London Mathematical Society 80 (2) (2000) 491–511.
[22] M. Shulman, Homotopy limits and colimits and enriched homotopy theory. ArXiv: math/0610194v2, 2006.
[23] R.M. Vogt, Homotopy limits and colimits, Mathematische Zeitschrift 134 (1973) 11–52.